DIFFERENTIAL EQUATIONS IN HILBERT-MUMFORD CALCULUS

ZIV RAN

ABSTRACT. An evolution-type differential equation encodes the intersection theory of tautological classes on the Hilbert scheme of a family of nodal curves.

INTRODUCTION

Let X/B be a family of nodal or smooth curves and L a line bundle on X. Let $X_B^{[m]}$ denote the relative Hilbert scheme of length-m subschemes of fibres of X/B, and $\Lambda_m(L)$ the tautological bundle associated to L, which is a rank-m bundle on $X_B^{[m]}$. The term 'Hilbert-Mumford Calculus' refers to the intersection calculus of 'tautological classes', i.e. polynomials in the Chern classes of $\Lambda_m(L)$. This calculus, which is an extension of the classical work of Macdonald [4], was developed in [5], [6] and other papers, where a number of examples and computations were given, with the more involved ones mostly based on the Macnodal computer program developed for this purpose by Gwoho Liu [3]. Our purpose here is to show that this calculus can be encoded in a linear second-order partial differential equation satisfied by a suitable generating function (see (2.35), (2.36) below). While the result is, in a sense, just a reformulation of results in [6], the advantages of the reformulation are that it uses the standard language of differential calculus and moreover avoids the recursiveness inherent in [6].

In more detail, let $W^m(X/B)$ denote the Hilbert scheme of length-m flags in fibres and consider the infinite-flag Hilbert scheme

$$W(X/B) = \varprojlim W^m(X/B) \subset \prod_m X_B^{[m]}$$

which is endowed with discriminant or big diagonal operators $\Gamma^{(m)}$ pulled back from $X_B^{[m]}$ and with classes L_i pulled back from the i-th X factor. It was shown in [5] that the Chern numbers of the tautological bundles can be expressed as linear combinations of monomials of the form (working left to right)

$$L_1^{a_1} L_2^{a_2} (\Gamma^{(2)})^{k_2} \ldots L_r^{a_r} (\Gamma^{(r)})^{k_r}.$$

Date: November 27, 2012.

1991 *Mathematics Subject Classification.* 14N99, 14H99.

Key words and phrases. Hilbert scheme, nodal curves, intersection theory.

Consequently we introduce the 'Hilbert potential'

$$G = \exp(\gamma\Gamma)\exp_\star(\sum \mu_i L^i)$$

in which $\star$ is external or 'Pontrjagin' product (whereas the 'implicit' or '.' product is intersection or, in the case of an operator like Γ, composition). Then the intersection calculus of [6] shows how to express G recursively in terms of elements of the so-called tautological module $T = T(X/B)$, and consequently how to read off numerical information. We show in Theorem 2.1 how to encode the latter into an equation in the γ- and μ_i-derivatives of G and its derivatives with respect to the 'space' variables corresponding to standard generators of T. This equation can be used to completely determine G.

In order to be able to express the appropriate relation in a familiar differential equation form, we introduce a formal model $\hat{T}$ for the tautological module T, essentially by replacing suitable generators by independent variables.

The use of differential equations to describe intersection theory associated to stable curves is not new. Our evolution equation is somewhat analogous to the 'quantum differential equation' of Gromov-Witten theory (see [1], Ch. 10 or [2], Ch. 28). Another well-known such equation is Witten's KdV equation, governing the intersection theory of the moduli space $\overline{\mathcal{M}}_g$ (see [7]). It would be interesting to find more direct connections.

1. Big tautologocal module

1.1. **Data.** We will fix a flat family X/B of nodal, possibly pointed, genus-g curves, which is 'split' in the sense that its boundary can be covered by finitely many projective families of the form $X^\theta/B(\theta) \to X/B$, each endowed with a pair of distinguished sections θ_x, θ_y called node preimages, that map to a node θ of X/B. We then have i-th boundary families X_i/B_i where

$$B_i = \coprod_{(\theta_1,...,\theta_i)} B(\theta_1, ..., \theta_i) = \coprod_{(\theta_1,...,\theta_i)} B(\theta_1) \times_B ... \times_B B(\theta_i)$$

(union over collections of i distinct nodes). This includes the case $i = 0$ where $B_0 = B$. To this we associate a *coefficient system*, in the form of a system of pairs of graded unital $\mathbb{Q}$-algebras

$$(A_{B_i} \to A_i) = \bigoplus (A_{B(\theta_1,...,\theta_i)} \to A_{(\theta_1,...,\theta_i)})$$

such that

(i) $(A_{B_i} \to A_i)$ admits a map to $(H^*(B_i, \mathbb{Q}) \to H^*(X_i, \mathbb{Q}))$;
(ii) Each $A = A_i$ contains an element ω_i that maps to $c_1(\omega_{X_i/B_i})$, plus elements that map to the distinguished sections, and each A_{B_i} contains elements mapping to Mumford classes and cotangent classes for the distinguished sections

(both those coming from X/B and node preimages). There are all compatible, e.g.

$$\omega_i|_{X^{\theta_1,\dots,\theta_i}} = \omega + \sum_{j=1}^{i}(\theta_{j,x} + \theta_{j,y}).$$

(iii) For any distinguished section σ over B_i, there is a pullback map $\sigma^* : A_i \to A_{B_i}$.

(iv) There are 'pullback' maps $(A_{B_i} \to A_i) \to (A_{B_{i+1}} \to A_{i+1})$ compatible with the various data. An element $\alpha \in A_i$ may be replaced by its image in $A_j, j > i$, whenever this makes sense.

1.2. **Generators, $\star$ product.** In [6] we defined the tautotological module

$$T = T_A(X/B) = \bigoplus T_A^m(X/B).$$

This is graded by the weight m which is the 'number of variables', i.e there is a canonical, not necessarily injective, map to the rational equivalence group

$$T^m \to A^\bullet_{\mathbb{Q}}(X_B^{[m]}).$$

T contains a 'classical' part T_0, which is a commutative algebra under external or Pontrjagin product (as distinct from intersection product), which will be denoted by $\star$. Via the correspondence

$$\begin{array}{ccccc} & & W^{m+m'}(X/B) & \to & X_B^{[m+m']} \\ (1.1) & \swarrow & & \searrow & \\ X_B^{[m]} & & & & X_B^{(m')} \end{array}$$

T is a module over T_0. A special role will be played by the diagonal classes of T_0: the monoblock diagonals

$$\Gamma_{(n)}[\alpha], \alpha \in A$$

and their $\star$- products, called polyblock diagonals. In fact, if we introduce a formal variable $t_n, n \geq 1$, we have a ring isomorphism

$$T_0 \simeq A_B[t_n A : n \in \mathbb{N}].$$

More concretely, T_0 is a direct sum of tensor products of symmetric powers of A over A_B, indexed by partitions.

In addition to polyblock diagonals, the tautological module also contains (iterated) node scrolls and node sections, of the form

$$F_j^n(\theta)[\gamma], Q_j^n(\theta)[\gamma], \gamma \in T_{A_\theta}(X^\theta/B(\theta))$$

(and their iterations). Thus, elements of the tautological module of X/B arise from analogous elements for a boundary family $X^\theta/B(\theta)$ via a node scroll $F_J^n(\theta)$ or a node section $Q_j^n(\theta)$. To describe iterated node/scroll sections systematically, let θ_F, θ_Q be mutually disjoint vectors of distinct nodes of X/B of respective dimensions b_F, b_Q,

and let j_F, n_F, j_Q, n_Q be vectors of natural numbers, indexed commonly with θ_F, θ_Q, respectively. Then we get iterated node classes

$$\begin{aligned}(1.2)\qquad & F^{n_F}_{j_F}(\theta_F)Q^{n_Q}_{j_Q}(\theta_Q)[\prod_\star \Gamma_{(m_i)}[\alpha_i]] \\ & = ...F^{n_{F,i}}_{j_{F,i}}(\theta_{F,i})...Q^{n_{Q,i}}_{j_{Q,i}}(\theta_{Q,i})...[\prod_\star \Gamma_{(m_i)}[\alpha_i]] \in T^{|m.|+b_F+b_Q}\end{aligned}$$

Thus via $F^{n_F}_{j_F}(\theta_F)Q^{n_Q}_{j_Q}(\theta_Q)[*]$, we get a map

$$T_0(X^{(\theta_F \amalg \theta_Q)}/B(\theta_F \coprod \theta_Q)) \to T(X/B).$$

F^n_j and Q^n_j are trivial unless $1 \le j < n$. $F^{n_F}_{j_F}(\theta_F)Q^{n_Q}_{j_Q}(\theta_Q)[\Gamma^R]$ is the class of the *closure* of the part of locus of type $F^{R_F}_{j_F}(\theta_F)Q^{R_Q}_{j_Q}(\theta_Q) \star \Gamma^R$ where the points in the factor corresponding to Γ^R are in the smooth part of X/B. It coincides with the class of the latter locus if either $R = 0$ or $R_Q = 0$, but differs from it otherwise. For example, the transfer formula of [6] reads, with this notation

$$\begin{aligned}(1.3)\qquad F^n_j(\theta) \star \Gamma_{(1)}[\alpha] &= F^n_j(\theta)[\Gamma_{(1)}[\alpha]], \\ Q^n_j(\theta) \star \Gamma_{(1)} &= Q^n_j(\theta)[\Gamma_{(1)}[\alpha]] + \theta^*(\alpha)F^{n+1}_j(\theta)\end{aligned}$$

In fact, a similar reasoning shows easily that

$$F^n_j(\theta) \star \Gamma_{(m)} = F^n_j(\theta)[\Gamma_{(m)}], \forall m \ge 1,$$

hence in fact

$$(1.4)\qquad F^n_j(\theta) \star \prod_\star \Gamma_{(m_i)}[\alpha_i] = F^n_j(\theta)[\prod_\star \Gamma_{(m_i)}[\alpha_i]],$$

The case of Q^n_j is more involved: the relationship is the following

Lemma 1.1. *Define rational numbers $r(n,j)^k_\ell$ for $0 < j < n$ by*

$$\begin{aligned}(1.5)\qquad & r(n,j)^j_{n+1} = 1; \\ & r(n,j)^k_\ell = \frac{1}{\ell - 1}((\ell - k)r^k_{\ell-1} + kr(n,j)^{k-1}_{\ell-1}), \ell > n+1; \\ & r(n,j)^*_* = 0, \text{otherwise.}\end{aligned}$$

and set

$$(1.6)\qquad F(n,j,\theta,m,s) = \theta^*_x(s|_{A_\theta})\sum_k r(n,j)^k_{n+m}F^{n+m}_k(\theta), s \in A$$

Then

$$(1.7)\qquad Q^n_j(\theta) \star \Gamma_{(m)}[s] = Q^n_j(\theta)[\Gamma_{(m)}[s]] + F(n,j,\theta,m,s)$$

Proof. The case $m = 1$ is just (1.3). The general case is obtained by applying punctual transfer (cf. [6], §3.3) $m - 1$ times to the result of $\star\Gamma_{(1)}[s]$, using [6], Prop. 3.19. $\square$

Remark 1.2. Because θ_x, θ_y both map to θ, we have $\theta_x^* = \theta_y$. Therefore we may denote both by θ^* and write the map $s \mapsto F(n, j, \theta, m, s)$ as

$$F(n, j, \theta, m) = \theta^* \sum_k r(n, j)_{n+m}^k F_k^{n+m}(\theta).$$

Also, $\theta_x^*(\omega) = 0$ (by residues), $\theta_x^*(1) = 1$ (trivially). $\square$

The same argument shows the following more general statement

Proposition 1.3. *We have, with the above notations,*

(1.8)

$$\begin{aligned}
&F_{j_F}^{n_F}(\theta_F) Q_{j_Q}^{n_Q}(\theta_Q) \star \Gamma_{(m_1)}[s_1] \star \dots \star \Gamma_{(m_r)}[s_r] = \\
&\quad F_{j_F}^{n_F}(\theta_F) Q_{j_Q}^{n_Q}(\theta_Q) [\Gamma_{(m_1)}[s_1] \star \dots \star \Gamma_{(m_r)}[s_r]] \\
&\quad + \sum_{i,j} F_{j_F}^{n_F}(\theta_F) F(n_{Q,i}, j_{Q,i}, \theta_{Q,i}, m_j, s_j) Q_{j_Q \setminus j_{Q,i}}^{n_Q \setminus n_{Q,i}}(\theta_Q \setminus \theta_{Q,i}) [\Gamma_{(m_1)}[s_1] \star \dots \widehat{\Gamma_{(m_j)}[s_j]} \dots \Gamma_{(m_r)}]
\end{aligned}$$

$\square$

Because F classes are represented by $\mathbb{P}^1$-bundles, they automatically have vanishing integrals, so a nice simple consequence of Proporsition 1.3 is

Corollary 1.4. *We have*

$$\begin{aligned}
\int F_{j_F}^{n_F}(\theta_F) Q_{j_Q}^{n_Q}(\theta_Q) \star \Gamma_{(m_1)}[s_1] \star \dots \star \Gamma_{(m_r)}[s_r] = \\
\int F_{j_F}^{n_F}(\theta_F) Q_{j_Q}^{n_Q}(\theta_Q) [\Gamma_{(m_1)}[s_1] \star \dots \star \Gamma_{(m_r)}[s_r]]. \quad \square
\end{aligned} \tag{1.9}$$

Note that the tautological module T splits naturally as

$$T = \bigoplus_{\theta.} T_{\theta.}$$

where the sum is over all vectors of distinct nodes and $T_{\theta.}$ consists of the classes that come from the $\theta.$ boundary via a node scroll/section construction (though $T_{\theta.}$ is independent of the ordering of $\theta.$, it is convenient to specify the ordering). Thus

$$T_{\theta.} = \bigoplus_{\substack{\theta. = \theta_F \coprod \theta_Q, \\ n_F, n_Q, j_F, j_Q}} F_{j_F}^{n_F}(\theta_F) Q_{j_Q}^{n_Q}(\theta_Q) T_{A^{\theta.}}(X^{\theta.}/B(\theta.)) \tag{1.10}$$

where $X^{\theta.}/B(\theta.)$ is the desingularized boundary family corresponding to $\theta.$, endowed with the node-preimage sections, and $A^{\theta.}$ is a coefficient ring on $X^{\theta.}/B(\theta.)$ as above.

1.3. **Standard model.** We describe a standard model, actually just a notation change, for the tautological module T. This will be a free module $\hat{T}$ over a power series ring $\hat{T}_0$, in which $F^*_*(*), Q^*_*(*)$ and $\Gamma_{(*)}[*]$ become variables or formal symbols. This will enable us to express the structure of T in terms of standard operations such as differential operators.

For each $n \geq 1$ let t_n be a formal variable, let $t_0 = 1$, and set

$$A\langle\infty\rangle = \bigoplus_{n=0}^{\infty} At_n, A\langle\infty\rangle_{(\theta.)} = \bigoplus_{n=0}^{\infty} A_{(\theta.)}t_n$$

as A_B or $A_{B(\theta.)}$-module, respectively. Then we have an A_B-algebra

$$\hat{T}_0 = A_B[A\langle\infty\rangle].$$

We think of generators $\alpha t_n \in A\langle\infty\rangle$ as corresponding to $\Gamma_{(n)}[\alpha]$.and assign them weight n. Likewise,

$$\hat{T}_{0,(\theta.)} = A_{B(\theta.)}[A\langle\infty\rangle_{(\theta.)}].$$

We set

$$\hat{T}_{0,*} = \bigoplus_{(\theta.)} T_{0,(\theta.)}, \hat{T}_{0,i} = \bigoplus_{|(\theta.)|=i} T_{0,(\theta.)}.$$

For each node θ, we designate formal variables $\phi^n_j(\theta), \chi^n_j(\theta)$ corresponding to the node classes $F^n_j(\theta), Q^n_j(\theta)$. Now let θ_ϕ, θ_χ be disjoint collections of distinct nodes, and let $n_\phi, j_\phi, n_\chi, j_\chi$ be correspondingly-indexed vectors of natural numbers. Then set

$$\phi^{n_\phi}_{j_\phi}(\theta_\phi)\chi^{n_\chi}_{j_\chi}(\theta_\chi) = \prod_{(n_\phi, j_\phi, \theta_\phi)} \phi^n_j(\theta) \prod_{(n_\chi, j_\chi, \theta_\chi)} \chi^n_j(\theta),$$

$$\hat{T} = \bigoplus \phi^{n_\phi}_{j_\phi}(\theta_\phi)\chi^{n_\chi}_{j_\chi}(\theta_\chi)\hat{T}_{0,\theta_\phi \coprod \theta_\chi}$$

Thus, $\hat{T}$ is generated by symbols of the form

$$\phi^{n_\phi}_{j_\phi}(\theta_\phi)\chi^{n_\chi}_{j_\chi}(\theta_\chi)\prod(t_{n_i}\alpha_i), \alpha_i \in A_{\theta_\phi \coprod \theta_\chi}$$

and is a direct sum of $A_{(\theta.)}$ modules for the various collections $(\theta.)$ of distinct nodes. Moreover $\hat{T}$ is a $\hat{T}_{0,*}$-module.

Note the map

$$h : \hat{T} \to T$$

$$h(\phi^{n_\phi}_{j_\phi}(\theta_\phi)\chi^{n_\chi}_{j_\chi}(\theta_\chi)\prod(t_{n_i}\alpha_i)) = F^{n_\phi}_{j_\phi}(\theta_\phi)Q^{n_\chi}_{j_\chi}(\theta_\chi) \star \prod_{\star} \Gamma_{(n_i)}[\alpha_i] \tag{1.11}$$

h is a bijection under which the $T_{0,*}$-module structure corresponds to $\star$ multiplication.

Remark 1.5. Note that for any A_B-linear map $\psi : A \to A$, there is a derivation $\psi t_n \partial / \partial t_n$ of T defined by

$$\psi t_n \partial / \partial t_n (t_m \alpha) = \begin{cases} \psi(\alpha), & m = n, \\ 0, & m \neq n; \end{cases} \tag{1.12}$$

$$\psi t_n \partial / \partial t_n (\phi^*_*(*)||\chi^*_*(*)) = 0.$$

Similarly, if $\psi : A \to A_{B_1}$ is an A_B-linear map, we can define a derivation

$$\psi \phi^n_j(\theta) \partial / \partial t_n : \hat{T} \to \hat{T},$$

$$\psi \phi^n_j(\theta) \partial / \partial t_n (t_m \alpha) = \begin{cases} \psi(\alpha)\phi^n_j(\theta), & m = n, \\ 0, & m \neq n; \end{cases} \tag{1.13}$$

$$\psi \phi^n_j(\theta) \partial / \partial t_n (\phi^*_*(*)||\chi^*_*(*)) = 0.$$

Remark 1.6. Though not critical for our purposes, $\hat{T}$ can be made into a commutative associative ring under the proviso that $\phi|\chi$ monomials must involve only distinct nodes θ: i.e.

$$(\phi|\chi)^*_*(\theta)(\phi|\chi^*_*)(\theta) = 0;$$

otherwise (i.e. where distinct θs are involved) ϕs and χs multiply formally.

1.4. Γ **action.** For enumerative purposes, a crucial feature of T is the weight-graded action by the discriminant Γ. The nonclassical (boundary) part of the action is described by the following rules.

$$(\Gamma . \prod\nolimits^\star \Gamma_{(n_i)}[\alpha_i])_\theta = \sum_i \sum_{0<j<n_i} \frac{j(n_i - j)n_i}{2} F^{n_i}_j(\theta)[\prod_{i' \neq i}\nolimits^\star \Gamma_{(n_{i'})}[\alpha_{i'}]] \tag{1.14}$$

(1.15)

$$-\Gamma.(F^n_j(\theta)[\gamma]) = Q^n_j(\theta)[\gamma] + F^n_j[e^n_{j+1}.\gamma], \gamma \in T_{A^\theta}(X^\theta/B(\theta))$$

$$e^n_j(\theta) = -\Gamma_{X^\theta/B(\theta)} - (n-j+1)i(\theta_x) - ji(\theta_y) + \binom{n-j+1}{2}\psi_x(\theta) + \binom{j}{2}\psi_y(\theta)$$

$$-\Gamma.Q^n_j(\theta)[\gamma] = Q^n_j[e^n_j(\theta).\gamma]. \tag{1.16}$$

Here $i(\theta_{x|y})$ refers to interior multiplication (see §1.5).

The classical or interior part of the action of Γ on T_0 is described by

$$\Gamma_0(\prod\nolimits^\star \Gamma_{(n_j)}[\alpha_j]) = \sum_{j<j'} \Gamma_{(n_j+n_{j'})}[\alpha_j.\alpha_{j'}] \prod_{k \neq j,j'} \Gamma_{(n_k)}[\alpha_k] - \sum \binom{n_j}{2} \Gamma_{n_j}[\omega\alpha_j] \tag{1.17}$$

Note that Γ_0 has the nature of a second-order differential operator, in the following sense. let F be $\star$- polynomial in the $\Gamma_{(n)}[\alpha]$ with coefficients in A_B. Let $\hat{F} = h^{-1}(F) \in \hat{T}$, i.e. $\hat{F}$ is the result of plugging in $t_n\alpha$ for each $\Gamma_{(n)}[\alpha]$ (and replacing $\star$ product by

ordinary product). For $\alpha \in A$, let $\alpha\,\partial\,/\,\partial\,t_n$ be the unique A_B-derivation on $\hat{T}_0$ such that

$$\alpha\,\partial\,/\,\partial\,t_n(\alpha' t_{n'}) = \begin{cases} \alpha\alpha', n' = n \\ 0, n' \neq n. \end{cases}$$

and of course $\partial\,/\,\partial\,t_n = 1_A\,\partial\,/\,\partial\,t_n$. Then

$$\Gamma_0 F = h(\hat{\Gamma}_0 \hat{F}), \quad \text{where}$$

$$\hat{\Gamma}_0 := \sum_{n \leq n'} n n' t_{n+n'} \frac{\partial^2}{\partial\,t_n\,\partial\,t_{n'}} - \sum_n \binom{n}{2} t_n \omega \frac{\partial}{\partial\,t_n}. \tag{1.18}$$

For example,

$$\hat{\Gamma}_0((t_n\alpha)(t_{n'}\alpha')) = nn' t_{n+n'}(\alpha.\alpha') - \binom{n}{2} t_n(\omega.\alpha)(t_{n'}\alpha') - \binom{n'}{2}(t_n\alpha)t_{n'}(\omega.\alpha'), n \neq n'.$$

This will be amplified below.

For later reference, we note the relation between Γ_0 on $T_0(X/B)$, as given by (1.17), and the corresponding operator on $T_0(X^\theta/B(\theta))$ for a boundary family $X^\theta/B(\theta)$. The only difference is that $\omega = \omega_{X/B}$ is replaced by $\omega_{X^\theta/B(\theta)} = \omega(-\theta_x - \theta_y)$, where θ_x, θ_y are the node preimage sections. Consequently, if we let $i^{(2)}$ be the derivation with respect to $\star$ product defined by

$$i^{(2)}(\sigma)\Gamma_{(n)}[\alpha] = \binom{n}{2}\Gamma_{(n)}[\sigma.\alpha]. \tag{1.19}$$

then we have

$$\Gamma_{X^\theta/B(\theta),0} = \Gamma_{0,X/B}|_{X^\theta} + i^{(2)}(\theta_x + \theta_y) \tag{1.20}$$

1.5. **Interior multiplication.** Given any class $\alpha \in A$, there is an interior multiplication action $i(\alpha)$ on the tautological module T: this is determined by the following conditions (where we recall that a node θ is viewed as a map $B(\theta) \to X$ and yields a pullback $\theta^* : A \to A_{B(\theta)}$):

(i) $i(\alpha)$ is a derivation with respect to $\star$ product;
(ii) $i(\alpha)\Gamma_{(n)}[\beta] = n\Gamma_{(n)}[\alpha.\beta]$;
(iii) $i(\alpha)F^n_j(\theta)[\beta] = F^n_j(\theta)[i(\alpha)\beta] + (\theta^*(\alpha))F^n_j(\theta)[\beta]$
(iv) $i(\alpha)Q^n_j(\theta)[\beta] = Q^n_j(\theta)[i(\alpha)\beta] + (\theta^*(\alpha))Q^n_j(\theta)[\beta]$.

In applications, α will usually be a section (hence disjoint from the node θ), so the second summand in the last two formulas it trivial. Therefore in such cases $i(\alpha)$ corresponds in the model $\hat{T}$ to the operator

$$\delta(\alpha) := \sum_n n t_n \alpha\,\partial\,/\,\partial\,t_n. \tag{1.21}$$

Similarly, the operator $i^{(2)}(\alpha)$ defined above corresponds to the derivation

$$\delta^{(2)}(\alpha) = \sum_n \binom{n}{2} t_n \alpha\, \partial / \partial t_n. \tag{1.22}$$

1.6. S**- transformation.** We seek a transformation on the tautological module taking $Q_j^n[\alpha]$ to $Q_j^n \star \alpha$. To this end, define rational numbers $r(n,j)_\ell^k$ as in (1.6) (see Remark 1.2). Then set, as in (1.12)

$$\phi(n,j,\theta,m) = \sum_k r(n,j)_{n+m}^k \phi_k^{n+m}(\theta)\theta^* \tag{1.23}$$

$$\hat{S} = \sum \phi(n,j,\theta,m) \frac{\partial^2}{\partial \chi_j^n(\theta)\, \partial t_m} \tag{1.24}$$

Then Proposition 1.3 shows that $\hat{S}$ corresponds to an operator S on T such that

$$Q_j^n(\theta)[\Gamma_{(m)}[s]] = Q_j^n(\theta) \star [\Gamma_{(m)}[s]] - SQ_j^n(\theta) \star \Gamma_{(m)}[s]$$

hence more generally

$$F_{j_F}^{n_F}(\theta_F) Q_{j_Q}^{n_Q}(\theta_Q)[\prod_\star \Gamma_{(m.)}[s.]] = (I - S)\left(F_{j_F}^{n_F}(\theta_F) Q_{j_Q}^{n_Q}(\theta_Q) \star \prod_\star \Gamma_{(m.)}[s.] \right). \tag{1.25}$$

Note that $\hat{S}^{b+1} = 0$ where $b = \dim(B)$. Consequently,

$$(I - S)^{-1} = I + S + ... + S^b. \tag{1.26}$$

2. Evolution equation

To introduce our evolution equation, we need some notation. First recall the corresponding to $\alpha \in A$ (see (1.21), (1.22)):

$$\begin{aligned} \delta(\alpha) &= \sum_{n'} n t_n \alpha \frac{\partial}{\partial t_n} \\ \delta^{(2)}(\alpha) &= \sum_n \binom{n}{2} t_n \alpha \frac{\partial}{\partial t_n}. \end{aligned} \tag{2.27}$$

Then set

$$\delta_j^n(\theta) = -(n-j+1)\delta(\theta_x) - j\delta(\theta_y) + \binom{n-j+1}{2} \psi_x(\theta) + \binom{j}{2} \psi_y(\theta) \tag{2.28}$$

where the ψ terms refer to the appropriate multiplication operators. This is a first-order differential operator.

We will need to express the discriminant operator in terms of $\hat{T}$ with its $\hat{T}_0$-module structure, which will involve rewriting terms like $Q_j^n(\theta)[\Gamma_{(n)}]$ in terms of $Q_j^n(\theta) \star \Gamma_{(n)}$. To this end, let $\hat{\Gamma}$ be the operator on $\hat{T}$ corresponding to Γ. It is $\hat{\Gamma}$ whose powers we

wish to compute, as this will yields powers of Γ. The idea is to achieve that via a change of variable. Thus set, using the notation of §1.6,

$$\tilde{\Gamma} = (I-S)^{-1}\hat{\Gamma}(I-S). \tag{2.29}$$

Then via $\hat{\Gamma}^k = (I-S)\tilde{\Gamma}^k(I-S)^{-1}$, it suffices to compute powers of $\tilde{\Gamma}$. But $\tilde{\Gamma}$ is a relatively 'elementary': specifically, a second-order differential operator. In the above notations, we have, by a direct computation,

(2.30)

$$\begin{aligned}\tilde{\Gamma} =& \hat{\Gamma}_0 + \sum \frac{j(n-j)n}{2}\theta_x^*\phi_j^n(\theta)\frac{\partial}{\partial t_n} - \sum \chi_j^n(\theta)\frac{\partial}{\partial \phi_j^n(\theta)} \\ & - \sum \phi_j^n(\theta)(\delta_{j+1}^n(\theta) - \delta^{(2)}(\theta_x+\theta_y))\frac{\partial}{\partial \phi_j^n(\theta)} + \chi_j^n(\theta)(\delta_j^n(\theta) - \delta^{(2)}(\theta_x+\theta_y))\frac{\partial}{\partial \chi_j^n(\theta)}\end{aligned}$$

where $\theta_x^*\phi_j^n(\theta)\frac{\partial}{\partial t_n}$ is as in Remark 1.5. Here the $\delta^{(2)}(\theta_x+\theta_y)$ term comes from the difference between $\omega_{X/B}$ and $\omega_{X^\theta/B(\theta)}$. Notice that because S does not involve the t variables, $\hat{\Gamma}_0$ coincides with the 'pure- t' or classical portion of $\tilde{\Gamma}$.

Now we might consider the generating function $\exp(\gamma\tilde{\Gamma})$ which encodes information about the powers of the discriminant operator Γ (weight unspecified). As discussed in the Introduction, this is not sufficient for enumerative applications, which require monomials involving discriminants of different weights and external multiplications. Fortunately the extension is not difficult to obtain.

To this end let $\alpha_1, ..., \alpha_r \in A$ be a set of homogeneous elements. The results of [5] and [6] show that Chern numbers of tautological bundles $\Lambda_m(L)$, for a line bundle L on X, on the flag-Hilbert schemes $W^m(X/B)$ of nodal curve families X/B are given by linear combinations of monomials of the form (read left to right)

$$M = (\star\Gamma_{(1)}[\alpha_1])(\star\Gamma_{(1)}[\alpha_2])\Gamma^{k_2}...(\star\Gamma_{(1)}[\alpha_r])\Gamma^{k_r} \tag{2.31}$$

where $\alpha_i = L^{n_i}$. Accordingly, we define, extending the above,

$$G = \exp(\gamma\Gamma)\exp_\star(\sum \mu_i\Gamma_{(1)}[\alpha_i]) \in T[[\gamma, \mu_1, ..., \mu_r]], \tag{2.32}$$

let

$$\hat{G} = \exp(\gamma\hat{\Gamma})\exp(\sum \mu_i\alpha_i t_1) \in \hat{T}[[\gamma, \mu_1, ..., \mu_r]] \tag{2.33}$$

be the corresponding element, and

$$\tilde{G} = (I-S)^{-1}\hat{G}(I-S) \tag{2.34}$$

(see (1.26)). Note that the first exponential in (2.33) refers to composition of operators while the second refers to product in $\hat{T}_0$, which corresponds to $\star$ product. We will use integral for an element of $\hat{T}$ to denote the integral of the corresponding element of T.

Theorem 2.1. *The following differential equations hold:*

(2.35)

$$\partial\,\tilde{G}/\,\partial\,\gamma = \hat{\Gamma}_0\tilde{G} + \sum_{\theta,n,j} \frac{j(n-j)n}{2}\theta_x^*\phi_j^n(\theta)\,\partial\,\tilde{G}/\,\partial\,t_n - \sum_{\theta,n,j}\chi_j^n(\theta)\,\partial\,\tilde{G}/\,\partial\,\phi_j^n(\theta)$$

$$-\sum_{\theta,n,j}\phi_j^n(\theta)(\delta_{j+1}^n(\theta)-\delta^{(2)}(\theta_x+\theta_y))\,\partial\,\tilde{G}/\,\partial\,\phi_j^n(\theta)+\chi_j^n(\theta)(\delta_j^n(\theta)-\delta^{(2)}(\theta_x+\theta_y))\,\partial\,\tilde{G}/\,\partial\,\chi_j^n(\theta)$$

$$\partial\,\tilde{G}/\,\partial\,\mu_i = t_1\alpha_i\tilde{G} + \sum_{\theta,n,j}\theta^*(\alpha_i)\phi_j^{n+1}(\theta)\,\partial\,\tilde{G}/\,\partial\,\chi_j^n(\theta). \tag{2.36}$$

Moreover,

$$\int \phi_{j_\phi}^{n_\phi}(\theta_\phi)\chi_{j_\chi}^{n_\chi}(\theta_\chi)\prod t_{n_i}\alpha_i = \begin{cases} 0, n_\phi \neq \emptyset \\ \prod\int_X \alpha_i, n_\phi = \emptyset. \end{cases} \tag{2.37}$$

This Theorem, together with the obvious initial value $\tilde{G}(0,...,0) = 1$ enables the computation of $\tilde{G}$, hence of G, hence of monomials M as in (2.31).

Proof. To begin with, the first part of relation (2.37) is essentially obvious , as ϕ variables correspond to $\mathbb{P}^1$-bundles of type F. The second part follows from Corollary 1.4, as χ variables correspond to sections of type Q of the F-bundles, and as far as integrals are concerned, $Q[\alpha]$ is equivalent to $Q \star \alpha$.

Now the relation (2.35) encapsulates the computation of the Γ operator as carried out in [6], §2. Schematically, applying $(-\Gamma)$ to a class of the form $F[y]$, $F = F_j^n(\theta)$, yields the sum of

(i) the corresponding $Q[y]$ class;

(ii) a class $F[dy]$ where d is analogous to $\delta_{j+1}^n(\theta)$ above;

(iii) the class $F[-\Gamma y]$.

Applying $-\Gamma$ to $Q[y]$ yields a sum of only the last two types (with j in place of $j+1$).

The first and second terms on the right of (2.35) correspond to the interior and boundary part of applying Γ to polyblock diagonals and generally to the polyblock factor of an FQ- monomial as in (1.2) (see [6], Thm. 2.23). The third term represents item (i) above for the action of Γ on each F factor. In the final summation, the $\phi\delta$ term represents item (ii) above for each F, while the $\chi\delta$ term represents the corresponding term for each Q (see [6], Theorem 2.24 and Remark 2.26). The $\delta^{(2)}$ term are the result of 'ω adjustment' as in (1.20), i.e writing

$$\omega_{X^\theta/B(\theta)} = \omega_{X/B}\otimes\mathcal{O}_{X^\theta}(-\theta_x-\theta_y).$$

Because different nodes θ are disjoint, no products of θ-s appear.

Equation (2.36) is a consequence of the Transfer Theorem of [6] (see Theorem 3.4 and display (3.1.19)). The second term is a reflection of the $F_j^{n+1}(\theta)$ term in the transfer of $Q_j^n(\theta)$.

$\square$

REFERENCES

1. D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Math. Surveys and Monographs, vol. 68, Amer. Math. Soc, 1999.
2. K. Hori et al., *Mirror symmetry*, Clay math. monographs, vol. 1, Amer. Math. Soc., 2003.
3. Gwoho Liu, *The macnodal package for intersection theory on hilbert schemes of nodal curves*, web interface (small jobs only) at `http://gwoho.com/macnodal/index.html.a?format=html`; source + executable + instructions at `http://math.ucr.edu/~ziv/`.
4. I. G. MacDonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.
5. Z. Ran, *Geometry on nodal curves*, Compositio math **141** (2005), 1191–1212.
6. ———, *Tautological module and intersection theory on Hilbert schemes of nodal curves*, Asian J. Math. (2012), 1–64, `arxiv:0905.2229v5`.
7. E. Witten, *2-dimensional gravity and intersection theory on moduli space*, Surveys Differ. Geom. **1** (1991), 243–310.

MATH DEPT. UC RIVERSIDE
SURGE FACILITY, BIG SPRINGS ROAD,
RIVERSIDE CA 92521
E-mail address: `ziv.ran @ucr.edu`